

Are Vortex Numbers Preserved?

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Abstract

We study noncommutative vortex solutions that minimize the action functional of the Abelian Higgs model in 2-dimensional noncommutative Euclidean space. We first consider vortex solutions which are deformed from solutions defined on commutative Euclidean space to the noncommutative one. We construct solutions whose vortex numbers are unchanged under the noncommutative deformation. Another class of noncommutative vortex solutions via a Fock space representation is also studied.

1 Introduction

In the noncommutative Euclidean space, the instanton number is given by an integer which does not depend on the noncommutative parameter, for the instanton solutions given by ADHM construction [1, 2, 3, 4, 5]. Because of these observations, one can ask “Are topological charges unchanged when we deform the space from Euclidean space to noncommutative Euclidean space?”. To answer this question, we investigate a two dimensional Abelian Higgs model. Solutions of the Bogomol’nyi equations in this model are called vortex solutions, and the vortex solutions minimize the action functional of the Abelian Higgs model.

In this paper, we study vortex solutions in noncommutative Euclidean space. We consider solutions which are deformations of vortex solutions defined on commutative Euclidean space and ask if the vortex number changes under the noncommutative deformation. In this paper, we use Taubes’ solution [7] as the vortex solution before undergoing deformation. The main purpose of this paper is to show that vortex numbers of vortex solutions are unchanged under this noncommutative deformation.

The organization of this article is as follows. In the next section, we review some results about the two dimensional Abelian Higgs model and vortices, and we lay out

the notation of this article. In section 3, we define and discuss the noncommutative deformation of the Abelian Higgs model. In section 4, we investigate the noncommutative vortex solutions deformed from the commutative vortex solutions and their vortex numbers. Our main claim is that the vortex number is unchanged. At first, we show that the vortex number is unchanged under certain conditions. Next, we solve the noncommutative vortex equations, and we show that the solutions satisfy these conditions. In section 5, another type of solution is treated. These solutions are not given by deformations of commutative vortex solutions, but are constructed using the Fock space representation. We show that one of the solutions is given by a bounded function.

2 Taubes' Vortex Solutions

We summarize the $U(1)$ gauge theory in commutative \mathbb{R}^2 . The gauge theory is defined by an action functional invariant under the gauge transformation. For example, the gauge symmetry is defined by the Higgs field. Higgs field ϕ , a complex scalar field. Let G be the group of gauge transformations associated to $U(1)$. For $g \in G$, the gauge transformation is defined as

$$\phi \rightarrow g\phi.$$

Noting that $\partial_\mu \phi$ is not covariant under this gauge transformation, we introduce the covariant derivative operator by

$$\nabla_\mu := \partial_\mu - iA_\mu, \quad (2.1)$$

where A_μ are the components of a local 1-form (a section of the cotangent bundle on \mathbb{R}^2). Its gauge transformation is defined by

$$\mathbf{A} \rightarrow ig\mathbf{d}g^{-1} + \mathbf{A}. \quad (2.2)$$

Here $\mathbf{A} := A_\mu dx^\mu \in \Omega^1$. Under the gauge transformation,

$$\nabla_\mu \phi = \partial_\mu \phi - iA_\mu \phi \quad (2.3)$$

is covariant.

For later convenience, we introduce complex coordinates for \mathbb{R}^2 and A_μ . On \mathbb{R}^2 , we use the following complex coordinates ;

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2), \quad (2.4)$$

and define differential operators $\partial, \bar{\partial}$ by

$$\partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2) , \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2) , \quad (2.5)$$

and define complex gauge fields by

$$A = \frac{1}{\sqrt{2}}(A_1 - iA_2) , \quad \bar{A} = \frac{1}{\sqrt{2}}(A_1 + iA_2) . \quad (2.6)$$

The gauge transformations are

$$A \rightarrow ig\partial g^{-1} + A , \quad \bar{A} \rightarrow -i\bar{\partial}gg^{-1} + \bar{A} . \quad (2.7)$$

The curvature for the connection A is expressed in the coordinates z, \bar{z} as

$$F_{zz} = F_{\bar{z}\bar{z}} = 0 , \quad F_{z\bar{z}} = iF_{12} = \partial\bar{A} - \bar{\partial}A .$$

We define the magnetic field B by ¹

$$B := -iF_{z\bar{z}} .$$

Using this representation, the covariant derivatives of the Higgs fields are

$$D\phi = (\partial - iA)\phi , \quad \bar{D}\phi = (\bar{\partial} - i\bar{A})\phi , \quad (2.8)$$

$$D\bar{\phi} = \partial\bar{\phi} + i\bar{\phi}A , \quad \bar{D}\bar{\phi} = \bar{\partial}\bar{\phi} + i\bar{\phi}\bar{A} . \quad (2.9)$$

It is worth commenting on the order of the fields. In the commutative case, the order is irrelevant e.g. $\bar{\phi}A = A\bar{\phi}$, and so on. But $\bar{\phi}A \neq A\bar{\phi}$ in the noncommutative case. Therefore, we use above expression in (2.9).

The functional studied in this paper (the static energy functional for the 2+1 dimensional Abelian Higgs model [6]) is given by

$$S = \int d^2z \left\{ -\frac{1}{2}(F_{z\bar{z}})^2 + D\phi\bar{D}\bar{\phi} + \bar{D}\phi D\bar{\phi} + \frac{1}{2}(\phi\bar{\phi} - 1)^2 \right\} . \quad (2.10)$$

Here $d^2z = d^2x$. We can regard this functional as the action functional of 2 dimensional Abelian Higgs model ². S can be rewritten as

$$S = S_T + \int d^2z \left\{ 2\bar{D}\phi D\bar{\phi} + \frac{1}{2}(B + (\phi\bar{\phi} - 1))^2 \right\} , \quad (2.11)$$

$$S_T := \int \left[\frac{1}{2} \left\{ \mathbf{d}(i\phi\mathbf{d}_A\bar{\phi} - i(\mathbf{d}_A\phi)\bar{\phi}) \right\} + \mathbf{B} \right] . \quad (2.12)$$

¹ We can treat our solutions as the soliton solutions in the 2+1 dimensional theory. The static energy density of the gauge field is described by the magnetic field B .

²In the following, we do not distinguish the energy functional in 2+1 dimensional theory from the 2 dimensional action functional. For example, a static solution that minimizes the 2+1 dimensional energy functional is identified with a solution that minimizes the 2 dimensional action functional.

Here, $d_A = d - iA$ and $B = Bdx^1 \wedge dx^2$. S_T is a topological term. Therefore the vortex equations are given by

$$\bar{D}\phi = (\bar{\partial} - i\bar{A})\phi = 0, \quad B + \phi\bar{\phi} - 1 = 0. \quad (2.13)$$

Solutions of these Bogomol'nyi equations (2.13) minimize the energy functional. We call these equations vortex equations and their solutions are called vortex solutions. We list some facts concerning vortex solutions.

Theorem 2.1 (Taubes, [7]). *Let (A_0, ϕ_0) be a smooth solution of (2.13). The vortex number,*

$$N_0 := \frac{1}{2\pi} \int d^2x B_0, \quad (2.14)$$

is an integer equal to the winding number of $\lim_{|z| \rightarrow \infty} \phi_0$, where $B_0 := B(A_0)$. Therefore, if $N_0 \neq 0$ then $\lim_{|z| \rightarrow \infty} \phi_0$ must have a zero and $\arg \phi_0$ cannot be smooth.

We will focus on noncommutative deformations of this theorem in section 4.

To describe local expressions for the Higgs field near the zero points, let us introduce some symbols. Let (A_0, ϕ_0) be a smooth solution of (2.13). Define the zero set $Z(\phi_0)$ by

$$Z(\phi_0) = \{z \in \mathbb{C} \mid \phi_0(z) = 0\} \quad (2.15)$$

Theorem 2.2 (Taubes, [7]). *Let (A_0, ϕ_0) be a smooth, locally L^2 solution of (2.13) of vortex number N . Then there exist N points $\{z_1, \dots, z_N\}$ in \mathbb{C} , such that*

$$Z(\phi_0) = \{z_1, \dots, z_N\}. \quad (2.16)$$

There is a neighborhood of each z_a in which

$$\phi_0(z) = (z - z_a)^{n_a} h_a(z), \quad (2.17)$$

where n_a is the multiplicity of the point z_a in $\{z_1, \dots, z_N\}$, and $h_a(z)$ is a C^∞ , nonvanishing function.

Finally, we list the following useful formula.

Theorem 2.3 (Taubes, [7]). *Let (A_0, ϕ_0) be a smooth, finite action solution to the equations (2.13). Then for any $\epsilon > 0$, there exists $M(\epsilon) < \infty$ such that*

$$0 < \frac{1}{2}(1 - |\phi_0(x)|^2) < M(\epsilon)e^{-r(1-\epsilon)}, \quad (2.18)$$

where $r = |x|$.

From (2.18), the asymptotic behaviors of the (A_0, ϕ_0) for large radius r are given by

$$|\phi_0| \sim 1 - Ce^{-r(1-\epsilon)} \quad (2.19)$$

$$\begin{aligned} |\partial\phi_0| &\sim |\bar{\partial}\phi_0| \sim C' \frac{1}{r} \\ |A_0| &\sim C'' \frac{1}{r}. \end{aligned} \quad (2.20)$$

Here, C, C', C'' are some constants.

In the following, we investigate the noncommutative deformations of this theory. In particular, we will carefully discuss whether the vortex number is constant.

3 The Noncommutative Abelian Higgs Model

In this section, we deform the Abelian Higgs model introduced in the previous section via the Moyal product [8]. The vortex equations and their solutions are also deformed.

3.1 The Noncommutative $U(1)$ Gauge Transformation

At first, let coordinates of noncommutative Euclidean space \mathbb{R}_θ^2 be x^μ , $\mu = 1, 2$, with commutation relations

$$[x^\mu, x^\nu] = i\theta\epsilon^{\mu\nu}, \mu, \nu = 1, 2, \quad (3.1)$$

where $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, ($\epsilon^{12} = 1$) is an anti-symmetric tensor and θ is a parameter called the noncommutative parameter. There are several representations of \mathbb{R}_θ^2 . In this section, we use the Moyal product [8]. The Moyal product is defined as an integral form

$$f(x) * g(x) := \frac{1}{2\pi\theta} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(y)g(z)e^{2iS(x,y,z)/\theta} dy dz, \quad (3.2)$$

$$\text{where } S(x, y, z) = (x, Jy) + (y, Jz) + (z, Jx) \text{ and } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for a suitable class of functions on \mathbb{R}^2 (e.g. subclass of Schwarz functions). For our purpose of this paper to find asymptotic solutions of deformed vortex solutions, we consider the formal version of (3.2) as follows:

$$\begin{aligned} f(x) * g(x) &:= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta \epsilon^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f(x) \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta \epsilon^{\mu\nu} \overrightarrow{\partial}_\nu\right)^n g(x). \end{aligned}$$

Here $\overleftarrow{\partial}_\mu$ is a derivative operator for $f(x)$ and $\overrightarrow{\partial}_\nu$ is for $g(x)$. Though we are working on formal vortex solutions in section 4, it will be an interesting problem to consider nonformal vortex solutions.

Let us summarize the $U(1)$ gauge theory on \mathbb{R}_θ^2 . As in section 2, that is the Higgs field is ϕ and the gauge transformation group is G . For $g \in G$, gauge transformations are defined as

$$\phi \rightarrow g * \phi.$$

We should comment here that the noncommutative $U(1)$ gauge symmetry is itself deformed from the commutative case. Let $U(x, \theta) \in G$ and \bar{U} be the complex conjugate of U , where G is the gauge transformation group of $U(1)$, such that

$$U * \bar{U} = \bar{U} * U = 1 \quad (3.3)$$

We can expand U as $U(x, \theta) = \sum_{k=0} U_k(x) \theta^k$. Then the unitary equation (3.3) is equivalent to

$$\begin{aligned} U_0 \bar{U}_0 &= 1 \\ U_0 \bar{U}_1 + U_1 \bar{U}_0 + \frac{1}{2} (\partial U_0 \bar{\partial} \bar{U}_0 - \bar{\partial} U_0 \partial \bar{U}_0) &= 0 \\ &\vdots \\ \sum_{0 \leq l \leq m \leq p \leq k} \partial^{m-l} \bar{\partial}^l U_{p-m} \bar{\partial}^{m-l} \partial^l \bar{U}_{k-p} \frac{(-1)^l \theta^k}{l!(m-l)! 2^m} &= 0 \\ &\vdots \end{aligned}$$

One degree of freedom of U_k is determined by solving the above unitary equation, and then only one degree for each U_k is left for the gauge transformation parameter. When the expansion of ϕ is given by $\sum \phi_k \theta^k$, the gauge transformation for each ϕ_k is

$$\begin{aligned} \phi &\rightarrow \phi' = \sum_k \phi'_k \theta^k = U * \phi \\ \phi_k &\rightarrow \phi'_k = \sum_{0 \leq l \leq m \leq p \leq k} \partial^{m-l} \bar{\partial}^l U_{p-m} \bar{\partial}^{m-l} \partial^l \phi_{k-l} \frac{(-1)^l}{2^m l!(m-l)!}. \end{aligned} \quad (3.4)$$

Note that for ϕ_0 the gauge transformation is the same as the commutative $U(1)$ theory.

Let us define the covariant derivative operator by

$$\nabla_\mu = \partial_\mu - i A_\mu, \quad (3.5)$$

where A_μ is a local 1-form whose gauge transformation is defined by

$$\mathbf{A} \rightarrow ig * \mathbf{d} * g^{-1} + g * \mathbf{A} * g^{-1}. \quad (3.6)$$

From this gauge transformation, we find that

$$\nabla_\mu * \phi := \partial_\mu \phi - iA_\mu * \phi. \quad (3.7)$$

is covariant under gauge transformation.

In the complex coordinates $A = \frac{1}{\sqrt{2}}(A_1 - iA_2)$ and $\bar{A} = \frac{1}{\sqrt{2}}(A_1 + iA_2)$, the gauge transformations are

$$A \rightarrow ig * \partial g^{-1} + g * A * g^{-1}, \quad \bar{A} \rightarrow -i\bar{\partial}g * g^{-1} + g * \bar{A} * g^{-1} \quad (3.8)$$

The curvature components of the connection A are given by

$$\begin{aligned} F_{zz} &= F_{\bar{z}\bar{z}} = 0 \\ F_{z\bar{z}} &= iF_{12} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z - i[A_z, A_{\bar{z}}]_*, \end{aligned}$$

where $[A, B]_* := A * B - B * A$. The magnetic field (in the sense of 2+1 dimension model) is defined by

$$B := -iF_{z\bar{z}}. \quad (3.9)$$

Although we are using the same notation for the curvature as for the commutative \mathbb{R}^2 , in the following, we consider only the noncommutative \mathbb{R}^2 so the notation should be clear.

Using these complex coordinates, the covariant derivatives of the Higgs fields are

$$D * \phi = (\partial - iA) * \phi, \quad \bar{D} * \phi = (\bar{\partial} - i\bar{A}) * \phi, \quad (3.10)$$

$$D * \bar{\phi} = \partial \bar{\phi} + i\bar{\phi} * A, \quad \bar{D} * \bar{\phi} = \bar{\partial} \bar{\phi} + i\bar{\phi} * \bar{A}. \quad (3.11)$$

3.2 The Action Functional and The Vortex Equations

The action functional for the noncommutative Abelian Higgs model [6] is given by

$$S = \int d^2z \left\{ -\frac{1}{2}(F_{z\bar{z}})_*^2 + D * \phi * \bar{D} * \bar{\phi} + \bar{D} * \phi * D * \bar{\phi} + \frac{1}{2}(\phi * \bar{\phi} - 1)^2 \right\}. \quad (3.12)$$

As in the commutative case, S can be rewritten as

$$S = S_T + \int d^2z \left\{ 2\bar{D} * \phi * D * \bar{\phi} + \frac{1}{2}(B + (\phi * \bar{\phi} - 1))_*^2 \right\}, \quad (3.13)$$

$$S_T := \int \left[\frac{1}{2} \left\{ \mathbf{d}(i\phi * \mathbf{d}_A * \bar{\phi} - i(\mathbf{d}_A * \phi) * \bar{\phi}) \right\} + \mathbf{B} \right]. \quad (3.14)$$

S_T is a topological term.

Therefore the vortex equations are given by

$$\bar{D} * \phi = (\bar{\partial} - i\bar{A}) * \phi = 0, \quad B + \phi * \bar{\phi} - 1 = 0. \quad (3.15)$$

We call solutions of these equations noncommutative vortices or noncommutative vortex solutions. Some solutions in [11, 12, 13, 14, 15, 16, 17, 18] have been constructed by using the operator formalism. These are different from the solutions discussed in section 4.

The formal expansions of the fields are

$$\phi = \sum_{n=0}^{\infty} \theta^n \phi_n(z, \bar{z}), \quad A = \sum_{n=0}^{\infty} \theta^n A_n(z, \bar{z}). \quad (3.16)$$

The k -th order equations for (3.15) are

$$-i(\partial\bar{A}_k + \bar{\partial}A_k) + \phi_k\bar{\phi}_0 + \phi_0\bar{\phi}_k - \delta_{k0} + C_k(z, \bar{z}) = 0 \quad (3.17)$$

$$\bar{\partial}\phi_k - i\bar{A}_k\phi_0 - i\bar{A}_0\phi_k + D_k(z, \bar{z}) = 0. \quad (3.18)$$

Here $C_k(z, \bar{z})$ is the coefficient of θ^k in $-[A, \bar{A}]_* + \phi * \bar{\phi} - (\phi_k\bar{\phi}_0 + \phi_0\bar{\phi}_k)$, so $C_k(z, \bar{z})$ is a function of $\{A_i, \bar{A}_j, \phi_m, \bar{\phi}_n | 0 \leq i, j, m, n \leq k-1\}$. Similarly, $D_k(z, \bar{z})$ is the coefficient of θ^k in $-i\bar{A} * \phi - (-i\bar{A}_k\phi_0 - i\bar{A}_0\phi_k)$ and a function of $\{A_i, \bar{A}_j, \phi_m, \bar{\phi}_n | 0 \leq i, j, m, n \leq k-1\}$.

In particular in the case of $k=0$, (3.17) and (3.18) coincide with the commutative U(1) vortex equations (2.13) i.e., $\bar{D}\phi_0 = (\bar{\partial} - i\bar{A}_0)\phi_0 = 0$ and $B_0 + \phi_0\bar{\phi}_0 - 1 = 0$, where $B_0 = -i(\partial\bar{A}_0 - \bar{\partial}A_0)$.

In the region $\phi_0 \neq 0$, substituting (3.18) into (3.17) for A_k and \bar{A}_k , we get

$$\left\{ \frac{\partial\phi_0}{\phi_0^2}(\bar{\partial}\phi_k - i\bar{A}_0\phi_k + D_k) - \frac{1}{\phi_0}(\Delta\phi_k - i\partial\bar{A}_0\phi_k - i\bar{A}_0\partial\phi_k + \partial D_k) \right\} + \{c.c.\} + \phi_k\bar{\phi}_0 + \phi_0\bar{\phi}_k - \delta_{k0} + C_k = 0. \quad (3.19)$$

Here $\{c.c.\}$ is the complex conjugate of preceding terms and $\Delta = \partial\bar{\partial}$.

Setting

$$\varphi_k := \frac{\phi_k}{\phi_0} + \frac{\bar{\phi}_k}{\bar{\phi}_0} = 2\text{Re}\left(\frac{\phi_k}{\phi_0}\right) \quad \text{and} \quad d_k = \frac{D_k}{\phi_0}, \quad (3.20)$$

by (3.19), φ_k , d_k satisfy

$$(-\Delta + |\phi_0|^2)\varphi_k = E_k \quad (3.21)$$

where

$$E_k := -C_k + \partial d_k - \bar{\partial}d_k. \quad (3.22)$$

From (2.18), there exists a positive constant C such that

$$|D_1| < C \frac{1}{1+r^3}, \quad |C_1| < C \frac{1}{1+r^4}, \quad |E_1| < C \frac{1}{1+r^4}. \quad (3.23)$$

We will use (3.23) to prove some of our main theorems. But in the proofs the actual power of r is not important ³.

3.3 Preliminary Facts

As in section 2, the vortex number $N_0 := \frac{1}{2} \int B_0$ is a integer corresponding to the winding number of $\lim_{|z| \rightarrow \infty} \phi_0$.

Let (A_0, ϕ_0) be a smooth solution of (2.13). Define I_k and $w(\bar{z})$ by

$$I_k(z, \bar{z}) = \exp \left(\int \frac{1}{\phi_k} (D_k - i\bar{A}_k \phi_0) \right), \quad w(\bar{z}) = \frac{1}{2\pi} \int_{\mathcal{B}} \frac{\bar{A}_0}{\zeta - \bar{z}} d\zeta \wedge d\bar{\zeta}, \quad (3.24)$$

where \mathcal{B} is a closed disc in \mathbb{C} . Using I_k and $w(\bar{z})$, the following theorem is given as well as the Theorem 2.2.

Theorem 3.1. *Let $\{(A_i, \phi_i) ; 0 \leq i \leq k\}$ be a smooth solution of (3.15). Then $(e^{-w} I_k(z, \bar{z}) \phi_k(z, \bar{z}))$ is complex analytic, that is*

$$\bar{\partial} (e^{-w} I_k(z, \bar{z}) \phi_k(z, \bar{z})) = 0. \quad (3.25)$$

[Proof] We note that

$$\bar{\partial} (e^{-w} I_k(z, \bar{z}) \phi_k(z, \bar{z})) = e^{-w} ((\bar{\partial} I_k) \phi_k + I_k \bar{\partial} \phi_k - (\bar{\partial} w) I_k \phi_k). \quad (3.26)$$

By definition,

$$\bar{\partial} I_k = \left(\frac{D_k}{\phi_k} - i\bar{A}_k \frac{\phi_0}{\phi_k} \right) I_k, \quad \bar{\partial} w = i\bar{A}_0. \quad (3.27)$$

From (3.26) and (3.27), we get (3.25).

□

Note that e^{-w} is a non-vanishing function. The holomorphic function $\Omega(z) := e^{-w} I_k \phi_k$ has a finite number of zeros in any bounded set \mathcal{B} . In a neighborhood of each zero z_a , there is a nonvanishing function such that $\Omega(z) = (z - z_a)^{n_a} \Omega_a(z)$.

³ From a naive observation, we get $|D_1| < C \frac{1}{1+r^3}$, $|C_1| < C \frac{1}{1+r^2}$, $|E_1| < C \frac{1}{1+r^2}$. But we can constrain fields by choosing a gauge condition. For example, from the gauge condition $\text{Im } \phi_0 = 0$, we can derive (3.23). The following discussion holds for both cases.

Theorem 3.2. Let $\{(A_i, \phi_i) ; 0 \leq i \leq k\}$ be a smooth, locally L^2 solution of (3.15). There exist N points $\{z_1, \dots, z_N\}$ in \mathbb{C} , such that

$$Z(\phi_k) := \{z \in \mathbb{C} : I_k \phi_k(z) = 0\} = \{z_1, \dots, z_N\}. \quad (3.28)$$

There is a neighborhood of each z_a in which

$$I_k \phi_k(z) = (z - z_a)^{n_a} h_a(z), \quad (3.29)$$

where n_a is the multiplicity of the point z_a in $\{z_1, \dots, z_N\}$, and $h_a(z) = e^w \Omega_a$ is a C^∞ , nonvanishing function.

4 Vortex Number

In this section, we show that the vortex number is constant for vortex solutions that are given by noncommutative deformations of Taubes' vortex solutions.

4.1 Noncommutative Vortex Number

We first study conditions which preserve the vortex number under a noncommutative deformation.

Theorem 4.1. If the vortex number of a classical solution (2.13) is $\frac{1}{2\pi} \int d^2x B_0 = N_0$ and $|\phi_k| < Cr^{-\epsilon}$, $|\partial_r \phi_k| < Cr^{-\epsilon+1}$, for some $\epsilon > 0$ and large r , then

$$\frac{1}{2\pi} \int d^2x B = N_0. \quad (4.1)$$

[Proof] Let F_k be the coefficient of θ^k in F_{12} . Then we have for $k > 0$

$$\begin{aligned} \int d^2x F_k &= -i \int d^2x (\partial \bar{A}_k - \bar{\partial} A_k) - [A, \bar{A}]_*|_k \\ &= \oint \mathsf{A}_k \\ &\quad \int \sum_{l+m+n=k, n \geq 1} \left\{ A_l \left(\bar{\partial} \frac{1}{2} \vec{\partial} - \bar{\partial} \frac{1}{2} \vec{\partial} \right)^n \frac{1}{n!} \bar{A}_m - \bar{A}_m \left(\bar{\partial} \frac{1}{2} \vec{\partial} - \bar{\partial} \frac{1}{2} \vec{\partial} \right)^n \frac{1}{n!} A_l \right\} \\ &= \oint \frac{1}{i\phi_0} (\bar{\partial} \phi_k - \bar{A}_0 \phi_k + D_k) + c.c. \\ &\quad -i \oint \sum_{l+m+n=k, n \geq 1} \left\{ A_l \left(\bar{\partial} \frac{1}{2} \vec{\partial} - \bar{\partial} \frac{1}{2} \vec{\partial} \right)^{(n-1)} \frac{1}{n!} d\bar{A}_m \right. \\ &\quad \left. - \bar{A}_m \left(\bar{\partial} \frac{1}{2} \vec{\partial} - \bar{\partial} \frac{1}{2} \vec{\partial} \right)^{(n-1)} \frac{1}{n!} dA_l \right\}. \end{aligned} \quad (4.2)$$

From the following facts, we get the result that we want.

$$|D_k| \leq C \frac{1}{r^{2+\epsilon}}, \quad |\partial \bar{A}_{k-1} \partial \phi_0| \leq \frac{1}{r^{2+\epsilon}} \quad (4.3)$$

$$|\bar{A}_0 \phi_k| \leq C \frac{1}{r^{1+\epsilon}} \quad (4.4)$$

$$|\bar{\partial} \phi_k| \leq C \frac{1}{r^{1+\epsilon}}, \quad A_k \leq C \frac{1}{r^{1+\epsilon}}, \quad (4.5)$$

where C is a constant. We use (3.17), (3.18) and (3.23) here. Then, $\int d^2x F_k = 0$.

□

We next show the following theorem.

Theorem 4.2. *Let $\phi_k, A_k, D_k, C_k, E_k$ be fields and functionals defined above. $\phi_k = O(r^{-\alpha_k})$, $A_k = O(r^{-\beta_k})$, $D_k = O(r^{-\delta_k})$, $C_k = O(r^{-\gamma_k})$ and $E_k = O(r^{-\eta_k})$, where $\alpha_k = 2k$, $\beta_k = 2k + 1$, $\gamma_k = 2k + 2$, $\delta_k = 2k + 1$ and $\eta_k = 2k + 2$ for $k \in \mathbb{Z}_{>0}$ ⁴.*

[Proof] The proof is by induction.

- (I) From asymptotic behaviors (2.19) and (2.20) and the vortex equations (3.17) and (3.18), for $k = 1$ we get $\alpha_1 = 2$, $\beta_1 = 3$, $\gamma_1 = 2$, $\delta_1 = 3$ and $\eta_1 = 2$.
- (II) Assume above the theorem for $k = 1, \dots, j-1$. By the definition of D_k , there exists a positive constant C such that

$$|D_j| < C \left\{ \sum_{i=1}^{j-1} \frac{1}{r^{(\alpha_{j-i}+\beta_i)}} + \sum_{n=1}^j \sum_{i=0}^{j-n} \frac{1}{r^{(\alpha_{j-i-n}+\beta_i+2n)}} \right\} = O\left(\frac{1}{r^{2j+1}}\right). \quad (4.6)$$

Therefore, $\delta_j = 2j + 1$. With this result for δ_j , we can prove the statements for α_k , β_k , γ_k and η_k , by similar arguments.

□

4.2 The Schrödinger equation and Vortex Solutions

To show that there exists a unique noncommutative vortex solution deformed from the Taubes' vortex solution, we consider the stationary Schrödinger equation

$$(-\Delta + V(x))u(x) = f(x) \quad (4.7)$$

⁴ Note that without the gauge fixing condition $\text{Im } \phi_0 = 0$, we can easily derive $\gamma_k = 2k$ and $\eta_k = 2k+2$.

in \mathbb{R}^2 , where $V(x)$ is a real valued C^∞ function. Throughout this section, we impose the following assumptions for $V(x)$

$$(a1) \quad V(x) \geq 0, \quad \forall x \in \mathbb{R}^2 \quad (4.8)$$

$$(a2) \quad \text{There exist } K \subset \mathbb{R}^2 \text{ and } \exists c > 0 \text{ such that } K \text{ is a compact set and} \\ \text{for } x \in \mathbb{R}^2 \setminus K, \quad V(x) \geq c \quad (4.9)$$

$$(a3) \quad \text{There exist } x_1, \dots, x_N \in \mathbb{R}^2 \text{ such that } V(x_i) = 0, V(x) > 0 \\ \text{for } x \notin \{x_1, \dots, x_N\} \quad (4.10)$$

$$(a4) \quad \text{For any } \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2, \text{ There exists a positive constant } C_\alpha \\ \text{such that } |\partial_x^\alpha(V - c)| \leq C_\alpha \text{ for any } x \in \mathbb{R}^2 \quad (4.11)$$

We note that the system (3.21) satisfies the assumptions (a1) – (a4). We set

$$H_l(n) := \{f \mid \|f\| := \sup_{x \in \mathbb{R}^2} (1 + |x|^n) |\partial_x^\alpha f(x)| < \infty \text{ for any } |\alpha| \leq l\} \quad (4.12)$$

for $n \in \mathbb{Z}_+$. We let C, C_α , etc. denote unimportant positive constants whose value may change from line to line unless otherwise stated. The next theorem's proof follows a series of lemmas.

Theorem 4.3. *Under the assumptions (a1) – (a4), there exists a unique solution $u \in H_l(n)$ of (4.7) for any $f \in H_l(n)$.*

Following Theorem 2.1 (iii), Theorem 3.3 and Theorem 3.8 in [9], we have

Lemma 4.4. *Under the assumptions (a1) – (a4), V is subcritical, i.e. There exists a positive solution $G(x, y)$ of*

$$(-\Delta + V(x))G(x, y) = \delta^2(x - y). \quad (4.13)$$

Consider the stationary Schrödinger equation

$$(-\Delta + c)u(x) = f(x) \quad (4.14)$$

in \mathbb{R}^2 , where c is a positive constant. The Green's function $G_c(x, y)$ for $(-\Delta + c)$ is given explicitly by

$$G_c(x, y) = \frac{1}{2\pi} K_0(\sqrt{c}|x - y|) = \int_0^\infty \frac{\cos(\sqrt{c}|x - y|t)}{\sqrt{t^2 + 1}} dt \quad (4.15)$$

where $K_0(z)$ is the modified Bessel function, with known asymptotic behavior (cf.[9])

$$\begin{aligned} K_0(z) &\sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad \text{for } |z| \gg 1 \\ K_0(z) &\sim \log|z| \quad \text{for } 0 < |z| \ll 1. \end{aligned} \quad (4.16)$$

Let us estimate the behavior of the Green's functions in (4.13) at large and small $|x - y|$.

Lemma 4.5. *Assume (a1) – (a4). For $|x - y| \leq r_0$ ($0 < r_0 < 1$), there exist constants C_1 and C_2 such that*

$$-C_1 \log |x - y| \leq G(x, y) \leq -C_2 \log |x - y|. \quad (4.17)$$

The proof of this lemma is given in [10] (cf. Theorem 4.2 in [9]).

Lemma 4.6. *Assume (a1) – (a4). Let r_1 be the radius of a disk \mathcal{D}_1 centered at the origin and with $K \subset \mathcal{D}_1$. For $|x - y| \geq r_1$, there exists a constant C such that*

$$G(x, y) \leq CG_c(x, y). \quad (4.18)$$

[Proof] For $|x| \geq r_1$,

$$(-\Delta + V(x))G_c(x, y) = (V(x) - c)G_c(x, y) \geq 0, \quad (4.19)$$

where we use (4.16). Therefore $G_c(x, y)$ is a superharmonic function with respect to the $(-\Delta + V(x))$. Since $\mathcal{B}_1 = \partial\mathcal{D}_1$ is compact, there exists a positive constant C such that

$$G(x, y) \leq CG_c(x, y) \text{ for } x \in \mathcal{B}_1. \quad (4.20)$$

By the maximal principle we get

$$G(x, y) \leq CG_c(x, y) \text{ for } |x - y| \geq r_1. \quad (4.21)$$

□

Now, using Lemmas 4.4-4.6, we show Theorem 4.3.

[Proof of Theorem 4.3]

To show $u \in H_l(n)$, we estimate $(1 + |x|^n)u(x)$. It is enough to consider the case $|x| \geq r_0$ or the fixed r_0 . In this case,

$$\begin{aligned} (1 + |x|^n)u(x) &= \int (1 + |x|^n)G(x, y)f(y)dy \\ &= \int_{|x-y| \leq r_0} (1 + |x|^n)G(x, y)f(y)dy \end{aligned} \quad (4.22)$$

$$+ \int_{|x-y| \geq r_1} (1 + |x|^n)G(x, y)f(y)dy \quad (4.23)$$

$$+ \int_{r_0 \leq |x-y| \leq r_1} (1 + |x|^n)G(x, y)f(y)dy \quad (4.24)$$

(I) Estimation of (4.22)

$$\begin{aligned}
(4.22) &\leq (1 + |x|^n) \left| \int_{|x-y| \leq r_0} G(x, y) f(y) dy \right| \\
&\leq C \int_{|x-y| \leq r_0} |G(x, y)| |(1 + |y|^4) f(y)| dy \\
&\leq C'' \int_0^{r_0} r \log r dr = C'''
\end{aligned} \tag{4.25}$$

Here we use the facts that there exists some constant C such that $1 + |x|^4 < C(1 + |y|^4)$ and we use Lemma 4.5.

(II) Estimation of (4.23)

$$\begin{aligned}
(4.23) &\leq C \int_{|x-y| \geq r_1} \frac{1}{2\pi} \sqrt{\frac{\pi}{2\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} (1 + |y|^n)^{-1} (1 + |y|^n) |f(y)| dy \\
&\leq C' \int_{|x-y| \geq r_1} \frac{1}{2\pi} \sqrt{\frac{\pi}{2\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} (1 + |y|^n)^{-1} dy.
\end{aligned} \tag{4.26}$$

Here we use (4.16). Let us introduce two subregions $\mathcal{A}_1(x, r_1, r_2) = \{y \in \mathbb{R}^2 \mid |x-y| \geq r_1, |y| \leq r_2, \text{ for fixed } x\}$ and $\mathcal{A}_2(x, r_1, r_2) = \{y \in \mathbb{R}^2 \mid |x-y| \geq r_1, |y| \geq r_2, \text{ for fixed } x\}$.

$$(4.26) = C' \left(\int_{\mathcal{A}_1} + \int_{\mathcal{A}_2} \right) \frac{1}{2\pi} \sqrt{\frac{\pi}{2\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} (1 + |y|^n)^{-1} dy. \tag{4.27}$$

We estimate the first term of (4.27).

$$\begin{aligned}
&\int_{\mathcal{A}_1} \frac{1}{2\pi} \sqrt{\frac{\pi}{2\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} (1 + |y|^n)^{-1} dy \\
&\leq C \int_{r_1}^{\infty} \frac{1}{\sqrt{c^{1/2}r}} (1 + r^n) e^{-\sqrt{c}r} r dr \leq C'.
\end{aligned} \tag{4.28}$$

Next we estimate the second term of (4.27).

$$\begin{aligned}
&\int_{\mathcal{A}_2} \frac{1}{2\pi} \sqrt{\frac{\pi}{2\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} (1 + |y|^n)^{-1} dy \\
&\leq C \int_{\mathcal{A}_2} \sqrt{\frac{1}{\sqrt{c}|x-y|}} e^{-\sqrt{c}|x-y|} \left(1 + \frac{|x-y|^n}{1 + |y|^n}\right) dy \\
&\leq C' \int_{r_1}^{\infty} \frac{1}{\sqrt{c^{1/2}r}} (1 + r^n) e^{-\sqrt{c}r} r dr \leq C''
\end{aligned} \tag{4.29}$$

(4.28) and (4.29) show that $(4.23) < C$.

(III) Existence of some constant C such that $(4.24) < C$ is trivial because the region of integration in (4.24) is compact.

Differentiating (4.7) sufficiently and using similar computations as above, we obtain the estimate for $(1 + |x|^n)|\partial_x^\alpha u| < \infty$ ($|\alpha| \leq l$).

From (I)-(III), we have Theorem 4.3.

□

Equation (3.21) is a particular example of (4.7), so Theorem 4.1 and 4.3 imply the following theorem.

Theorem 4.7. *Let A_0 and ϕ_0 be a Taubes' vortex solution stated in section 2, in other words, (A_0, ϕ_0) satisfy the equations (2.13) with the condition (2.18). Then there exists a unique solution (A, ϕ) of the noncommutative vortex equations (3.15) with $A|_{\theta=0} = A_0$, $\phi|_{\theta=0} = \phi_0$, and its vortex number is preserved:*

$$N = N_0, \text{ i.e. } \frac{1}{2\pi} \int d^2x B = \frac{1}{2\pi} \int d^2x B_0. \quad (4.30)$$

[Proof] Consider (4.7) with $V(x) = |\phi_0|^2$ and $f(x) = E_k$. From the facts in section 2, we find $V(x)$ satisfies (a1) – (a4). Next, we consider E_k . From (3.23), $E_1 \in H_\infty(4)$. If $E_i \in H_\infty(2i+2)$ ($i = 1, \dots, k-1$), as a result of Theorem 4.3, there exist unique solutions $\varphi_1, \dots, \varphi_{k-1}$. Then we find $E_k \in H_\infty(2k+2)$ from Theorem 4.2. Therefore $E_k \in H_\infty(2k+2)$ is proved for arbitrary k . Theorem 4.3 is applicable to (3.21) for arbitrary k , then it is shown that each φ_k is determined uniquely. Finally, Theorem 4.2 and Theorem 4.1 imply that $N = N_0$.

□

5 Noncommutative Vortex Solutions via the Fock Representation

Solutions of (3.15) are given in [11, 12, 13, 14, 15, 16], etc. These solutions are substantially different from the solution discussed in the previous section. The difference will be clear soon. In this section, we show the existence of bounded solutions via Fock space formalism. As a simple example, we investigate the properties of the solution in [11].

5.1 Fock space formalism

Using complex coordinates z_α , we introduce the following operators:

$$\hat{a} \equiv \frac{z}{\sqrt{\theta}}, \quad \hat{a}^\dagger \equiv \frac{\bar{z}}{\sqrt{\theta}}, \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0.$$

\hat{a}^\dagger is a creation operator and \hat{a} is an annihilation operator. We define a Hilbert space by

$$\mathcal{H} = \bigoplus \mathbb{C} |n\rangle, \quad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle, \\ \hat{a} |n\rangle = \sqrt{n} |n-1\rangle, \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle,$$

where $|n\rangle$ is a eigenvector of the number operator $\hat{n} \equiv \hat{a}^\dagger \hat{a}$, i.e. $\hat{n} |k\rangle = k |k\rangle$. An arbitrary operator has the following expression;

$$\hat{\mathcal{O}} = \sum_{n,m} \mathcal{O}_m^n |n\rangle \langle m|.$$

Differentiation is given by

$$\partial_\mu \hat{f}(\hat{x}) = [\hat{\partial}_\mu, \hat{f}(\hat{x})] = -i\theta^{-1} \epsilon_{\mu\nu} [\hat{x}^\nu, \hat{f}(\hat{x})].$$

Here $\hat{\partial}_\mu = -i\theta^{-1} \epsilon_{\mu\nu} \hat{x}^\nu$ and $\epsilon_{\mu\nu}$ is the inverse of $\epsilon^{\mu\nu}$, i.e. $\epsilon_{\mu\nu} \epsilon^{\nu\rho} = \delta_\mu^\rho$. In terms of \hat{a} , \hat{a}^\dagger , differentiation is expressed by

$$\partial \hat{f}(z, \bar{z}) = [\hat{\partial}, \hat{f}(\hat{x})] = -\frac{1}{\sqrt{\theta}} [\hat{a}^\dagger, \hat{f}(z, \bar{z})], \quad \bar{\partial} \hat{f}(\hat{x}) = [\hat{\bar{\partial}}, \hat{f}(z, \bar{z})] = \frac{1}{\sqrt{\theta}} [\hat{a}, \hat{f}(z, \bar{z})].$$

Integration is replaced by the trace operation,

$$\int d^2x f(x) = 2\pi\theta \text{Tr}_{\mathcal{H}} \hat{f}(\hat{x})$$

in the operator formalism.

The covariant derivative operator is defined by

$$\hat{\nabla}_\mu := \partial_\mu - i\hat{A}_\mu, \tag{5.1}$$

where \hat{A} is a gauge connection in the operator formalism. For a Higgs field $\hat{\phi}$ in the operator formalism, the covariant derivative is given by

$$\hat{\nabla}_\mu \hat{\phi} = [\hat{\partial}_\mu, \hat{\phi}] - i\hat{A}_\mu \hat{\phi} = -\hat{\phi} \hat{\partial}_\mu + (\hat{\partial}_\mu - i\hat{A}_\mu) \hat{\phi}, \tag{5.2}$$

where $\hat{\partial}_\mu = -i\theta^{-1}\epsilon_{\mu\nu}\hat{x}^\nu$.

The curvature is defined by

$$\hat{F}_{\mu\nu} = i[\hat{\nabla}_\mu, \hat{\nabla}_\nu] , \quad (5.3)$$

and the action functional of the gauge theory in noncommutative \mathbb{R}_θ^2 is given by

$$S_{gauge} = -2\pi\theta\frac{1}{2}\text{Tr}_{\mathcal{H}}\hat{F}_{\mu\nu}^2 . \quad (5.4)$$

5.2 An Explicit Solution

For a Higgs field $\hat{\phi}$ in the operator formalism, the covariant derivative has the complex expression :

$$\hat{D}\hat{\phi} := [\hat{\partial}, \hat{\phi}] - i\hat{A}\hat{\phi}, \quad \hat{\bar{D}}\hat{\phi} := [\hat{\bar{\partial}}, \hat{\phi}] - i\hat{\bar{A}}\hat{\phi} . \quad (5.5)$$

For \hat{B} the magnetic field in operator formalism, we have

$$\hat{B} = -i([\hat{\partial}, \hat{A}] - [\hat{\bar{\partial}}, \hat{A}] - [\hat{A}, \hat{\bar{A}}]) . \quad (5.6)$$

In this formulation, the vortex equations are

$$\hat{\bar{D}}\hat{\phi} = [\hat{\bar{\partial}}, \hat{\phi}] - i\hat{\bar{A}}\hat{\phi} = \frac{1}{\sqrt{\theta}}[\hat{a}, \hat{\phi}] - i\hat{\bar{A}}\hat{\phi} = 0 \quad (5.7)$$

$$\hat{B} + \hat{\phi}\hat{\bar{\phi}} - 1 = 0 \quad (5.8)$$

An explicit solution for (5.7) and (5.8) is given in [11] by

$$\hat{\phi} = \sum_{n=0}^{\infty} |n+1\rangle\langle n| , \quad \hat{A} = \frac{1}{i\sqrt{\theta}} \left(\hat{a} - \frac{\sqrt{\hat{n}}}{\sqrt{\hat{n}+1}} \hat{a} \right) . \quad (5.9)$$

This solution has topological charge $\theta\text{Tr}_{\mathcal{H}}\hat{B} = 1$. In [11], explicit solutions are given for arbitrary integer valued topological charge $\theta\text{Tr}_{\mathcal{H}}\hat{B} = n$. For simplicity, we discuss only (5.9).

We first translate the solution (5.9) into a $*$ product expression. $|n\rangle\langle m|$ can be rewritten as

$$\begin{aligned} |n\rangle\langle m| &= : \frac{\hat{a}^{\dagger n}}{\sqrt{n!}} e^{-\hat{a}^\dagger \hat{a}} \frac{\hat{a}^m}{\sqrt{m!}} : \\ &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{n!m!}\theta^{n+m}} \left(\frac{-1}{\theta} \right)^k \frac{1}{k!} \hat{z}^{k+m} \hat{\bar{z}}^{k+n} , \end{aligned} \quad (5.10)$$

where \sim : is normal ordering, which by definition moves all \hat{a} 's to the right of all the \hat{a}^\dagger 's. From this fact, the $*$ product expression of $|n\rangle\langle m|$ is given by

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{n!m!\theta^{n+m}}} \left(\frac{-1}{\theta}\right)^k \frac{1}{k!} z^{k+m} * \bar{z}^{k+n}. \quad (5.11)$$

Therefore the Higgs field in the solution (5.9) is

$$\phi = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(n!)^2 \sqrt{(n+1)\theta}} \left(\frac{-1}{\theta}\right)^k \frac{1}{k!} z^{k+n} * \bar{z}^{k+n+1} \quad (5.12)$$

$$= e^{\frac{\theta}{2}\partial\bar{\partial}} \varphi(z, \bar{z}), \quad (5.13)$$

where

$$\varphi(z, \bar{z}) := \bar{z} e^{-\frac{|z|^2}{\theta}} \sum_{n=0}^{\infty} \frac{1}{n! \theta^n \sqrt{(n+1)\theta}} |z|^{2n}. \quad (5.14)$$

By (5.14), this type of solution has a $1/\theta$ expansion, which differentiates solutions via Fock representation from the solutions in section 4. Let us prove the following theorem.

Theorem 5.1. $|\varphi| < \int dx \frac{\sqrt{\theta}}{x^2} (1 - e^{-\frac{x^2}{\theta}})$, where $|z| = x$.

[Proof]

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{n! \theta^n \sqrt{(n+1)\theta}} x^{2n+1}, \quad x \geq 0 \quad (5.15)$$

$$\begin{aligned} \frac{df(x)}{dx} - \frac{2x}{\theta} f(x) &= \frac{1}{\sqrt{\theta}} + \frac{3-2\sqrt{2}}{\sqrt{2}} \frac{x^2}{\theta\sqrt{\theta}} \\ &\quad + \sum_{n=1}^{\infty} \frac{(2n+3)\sqrt{n+1} - (2n+2)\sqrt{n+2}}{(n+1)!\sqrt{(n+1)(n+2)}\theta^{n+1}\sqrt{\theta}} x^{2(n+1)} \\ &< \frac{1}{\sqrt{\theta}} \left(1 + \frac{x^2}{2!\theta} + \sum_{n=1}^{\infty} \frac{(4n^2-4n+9)x^{2(n+1)}}{(n+1)!\sqrt{n+2}\theta^{n+1}8(n+1)^2(2n+1)} \right) \\ &< \frac{1}{\sqrt{\theta}} \left(1 + \frac{x^2}{2!\theta} + \frac{\theta}{x^2} \sum_{n=1}^{\infty} \frac{x^{2(n+1)}}{(n+2)!\theta^{n+2}} \right) \\ &= \frac{\sqrt{\theta}}{x^2} (e^{\frac{x^2}{\theta}} - 1). \end{aligned} \quad (5.16)$$

Then,

$$|\varphi(x)| = (e^{-\frac{x^2}{\theta}} f(x)) \leq \int dx \frac{\sqrt{\theta}}{x^2} (1 - e^{-\frac{x^2}{\theta}}). \quad (5.17)$$

□

This theorem shows that the existence of bounded solutions with expansions in $1/\theta$.

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